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Lattice Random Walks for Sets of Random Walkers. First Passage Times

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We have studied the mean first passage time for the first of a *set* of random walkers to reach a given lattice point on infinite lattices of D dimensions. In contrast to the well-known result of *infinite* mean first passage times for one random walker in all dimensions D, we find *finite* mean first passage times for certain well-specified sets of random walkers in all dimensions, except $D = 2$. The number of walkers required to achieve a finite mean time for the first walker to reach the given lattice point is a function of the lattice dimension D. For $D > 4$, we find that only one random walker is required to yield a finite first passage time, provided that this random walker reaches the given lattice point with unit probability. We have thus found a simple random walk property which "sticks" at $D > 4$.

KEY WORDS: Random walks; infinite lattices; first passage times.

1. INTRODUCTION

It is well known[1–5] that a walker executing a random walk of zero mean and finite variance on a one- or two-dimensional infinite lattice reaches any arbitrary lattice site with certainty. It is also known that the mean time to first reach a given site is infinite in both cases. In three and higher dimensions, there is a finite probability that a walker may never reach a given lattice site, i.e., there is a finite probability of escape.[1–5]

The above results pertain to the behavior of a single random walker. Very little is known about the behavior of a *set of random walkers*. In this paper we consider the extension of several well-known single-walker results to a set of N independent random walkers. We concentrate on the subset of the N walkers that does reach a given lattice site and study the statistical properties of the

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time for the first walker of this subset to reach the site. We find that the expected time of this first arrival has a very interesting behavior as a function of dimensionality.

In Section 2 we present some definitions and establish our notation. Section 3 contains our results. A summary and physical interpretations of these results are given in Section 4.

2. DEFINITIONS

Consider a symmetric random walk on an infinite, D-dimensional lattice, and let $p(\mathbf{j})$ be the probability that the walker makes a displacement $\mathbf{j}$ in one step. We define $P_n(\mathbf{l}|\mathbf{0})$ to be the probability that the random walker is at $\mathbf{l}$ after n steps, given that it was initially at the origin $\mathbf{0}$. The generating function for $P_n(\mathbf{l}|\mathbf{0})$ is defined as[2-4]

$$P(z;\mathbf{l}|\mathbf{0}) \equiv \sum_{n=0}^{\infty} z^n P_n(\mathbf{l}|\mathbf{0}) \tag{2.1}$$

and is given in D dimensions by

$$P(z;\mathbf{l}|\mathbf{0}) = \frac{1}{(2\pi)^D} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\exp(-i\mathbf{l}\cdot\boldsymbol{\theta})}{1 - z\lambda(\boldsymbol{\theta})}\, d^D\boldsymbol{\theta} \tag{2.2}$$

where the structure factor $\lambda(\boldsymbol{\theta})$ is defined as

$$\lambda(\boldsymbol{\theta}) \equiv \sum_{j_1} \cdots \sum_{j_D} p(\mathbf{j}) \exp(i\mathbf{j}\cdot\boldsymbol{\theta}) \tag{2.3}$$

We further let $f_n(\mathbf{l}|\mathbf{0})$ be the probability that the walker is at $\mathbf{l}$ *for the first time* after n steps, conditional on the initial position $\mathbf{l} = \mathbf{0}$. The generating function

$$f(z;\mathbf{l}|\mathbf{0}) = \sum_{n=0}^{\infty} z^n f_n(\mathbf{l}|\mathbf{0}) \tag{2.4}$$

is related to $P(z;\mathbf{l}|\mathbf{0})$ of Eq. (2.1) by[4]

$$f(z;\mathbf{l}|\mathbf{0}) = \frac{P(z;\mathbf{l}|\mathbf{0}) - \delta_{\mathbf{l}\mathbf{0}}}{P(z;\mathbf{0}|\mathbf{0})} \tag{2.5}$$

In particular, the quantity $f_{\mathbf{l}}$ defined by

$$f_{\mathbf{l}} \equiv \sum_{n=0}^{\infty} f_n(\mathbf{l}|\mathbf{0}) = \frac{P(1;\mathbf{l}|\mathbf{0}) - \delta_{\mathbf{l}\mathbf{0}}}{P(1;\mathbf{0}|\mathbf{0})} \leqslant 1 \tag{2.6}$$

is the probability that a walker starting from the origin will ever reach site $\mathbf{l}$ ($\mathbf{l} \neq \mathbf{0}$) or will ever return to the origin ($\mathbf{l} = \mathbf{0}$). Thus, if N independent random walkers start from the origin, then on the average only $f_{\mathbf{l}}N$ of them will subsequently reach site $\mathbf{l}$.

We shall be interested in the time that it takes random walkers *chosen from among those that do reach* **l** to arrive at that site. For this purpose we define $g_n(\mathbf{l}|\mathbf{0})$ as the conditional probability that a walker will step on **l** at step n for the first time, given that it will eventually arrive there. This probability is given by

$$g_n(\mathbf{l}|\mathbf{0}) = f_n(\mathbf{l}|\mathbf{0})/f_1 \tag{2.7}$$

The probability that a walker that eventually reaches **l** has not done so by step n then is

$$G_n(\mathbf{l}|\mathbf{0}) = \sum_{m=n+1}^{\infty} g_m(\mathbf{l}|\mathbf{0}) \tag{2.8}$$

The generating function for $G_n(\mathbf{l}|\mathbf{0})$ is

$$\begin{aligned} G(z;\mathbf{l}|\mathbf{0}) &= \sum_{n=0}^{\infty} z^n G_n(\mathbf{l}|\mathbf{0}) \\ &= \frac{1-(1/f_1)[P(z;\mathbf{l}|\mathbf{0})-\delta_{\mathbf{l}\mathbf{0}}]/P(z;\mathbf{0}|\mathbf{0})}{1-z} \end{aligned} \tag{2.9}$$

The sth moment of the first passage time distribution to **l** for a walker known to eventually reach **l** in D dimensions is given by

$$\langle n^s(\mathbf{l}|\mathbf{0})\rangle^{(D)} = \sum_{n=0}^{\infty} n^s g_n(\mathbf{l}|\mathbf{0}) = \sum_{n=0}^{\infty} [(n+1)^s - n^s] G_n(\mathbf{l}|\mathbf{0}) \tag{2.10}$$

In particular, the mean time for such a walker to reach **l** for the first time is

$$\langle n(\mathbf{l}|\mathbf{0})\rangle^{(D)} = \sum_{n=0}^{\infty} G_n(\mathbf{l}|\mathbf{0}) \tag{2.11}$$

Suppose we now observe N independent random walkers starting simultaneously from the origin and consider k of them chosen from among those that *do* reach **l**. The probability that none of the k has yet reached **l** by the nth step is $[G_n(\mathbf{l}|\mathbf{0})]^k$. The mean time for the first of the k walkers to reach **l** then is

$$\begin{aligned} \langle n(\mathbf{l}|\mathbf{0})\rangle_k^{(D)} &= \sum_{n=0}^{\infty} n\left[\sum_{q=1}^{k} \frac{k!}{(k-q)!\,q!}\, g_n^q(\mathbf{l}|\mathbf{0}) G_n^{k-q}(\mathbf{l}|\mathbf{0})\right] \\ &= \sum_{n=0}^{\infty} [G_n(\mathbf{l}|\mathbf{0})]^k \end{aligned} \tag{2.12}$$

The term in the brackets in the first part of (2.12) is a sum over the probabilities that 1, 2,..., k walkers arrive at **l** for the first time on the nth step.

In the next section we apply these formulas to random walks of finite variances

$$\sigma_i^2 \equiv \sum_{j_D=-\infty}^{\infty} \cdots \sum_{j_1=-\infty}^{\infty} j_i^2 p(\mathbf{j})$$

in one, two, three, and higher dimensions. For a nearest neighbor symmetric random walk on a cubic D-dimensional lattice, $\sigma_i^2 = 1/D$.

3. RESULTS

3.1. One Dimension

In order to study the convergence of the sum for $\langle n(\mathbf{l}|\mathbf{0})\rangle_k^{(1)}$ in (2.12), we need the asymptotic (large-n) behavior of $G_n(\mathbf{l}|\mathbf{0})$. This can be obtained from the analytic behavior of $G(z; \mathbf{l}|\mathbf{0})$ of Eq. (2.9) near $z = 1$ by using a Tauberian theorem. It has been shown[4] that for a symmetric walk $[p(j) = p(-j)]$, the behavior of $P(z; \mathbf{0}|\mathbf{0})$ near $z = 1$ is

$$P(z; \mathbf{0}|\mathbf{0}) \sim 1/\sigma[2(1-z)]^{1/2} \tag{3.1}$$

In one dimension the probability that a random walker eventually reaches any given site $\mathbf{l}$ is unity, i.e., $f_{\mathbf{l}} = 1$.[1,2] Use of (3.1) and the result[4]

$$P(z; \mathbf{0}|\mathbf{0}) - P(z; \mathbf{l}|\mathbf{0}) \sim |\mathbf{l}| + O(1-z) \tag{3.2}$$

in (2.9) then leads to[4]

$$G(z; \mathbf{l}|\mathbf{0}) \sim \sqrt{2}\,\frac{|\mathbf{l}|}{\sigma}\,\frac{1}{(1-z)^{1/2}} \tag{3.3}$$

near $z = 1$. The Tauberian theorem[6] allows us to assert that for large n

$$G_0 + G_1 + \cdots + G_n \sim \left(\frac{8n}{\pi}\right)^{1/2} \frac{|\mathbf{l}|}{\sigma} \tag{3.4}$$

and since the G_n are nonincreasing functions of n, we can thus infer that[6]

$$G_n \sim \left(\frac{2}{\pi}\right)^{1/2} \frac{|\mathbf{l}|}{\sigma}\,\frac{1}{\sqrt{n}} \tag{3.5}$$

From this result and Eq. (2.12), it therefore follows that the mean time for the first walker to reach $\mathbf{l}$ is finite if at least *three* random walkers start from the origin. More generally, it can easily be seen that the sth moment of the first passage time distribution for the first of the walkers to reach $\mathbf{l}$ is finite if there are at least $2s + 1$ random walkers starting from the origin. It can be shown that these conclusions are also valid if each of the random walkers starts at a different point.

It is difficult to actually evaluate the mean time $\langle n(\mathbf{l}|\mathbf{0})\rangle_3^{(1)}$ for the first of the three walkers to arrive at $\mathbf{l}$. However, it is easy to obtain the continuum analog of this result by starting from the diffusion equation

$$\frac{\partial P}{\partial t}(x, t) = \mathscr{D}\frac{\partial^2 P(x, t)}{\partial x^2} \tag{3.6}$$

subject to the initial condition $P(x, 0) = \delta(x)$. We wish to compute the mean first passage time to $x = A$ of the earliest of k independent diffusing particles to arrive there. For this purpose we must solve (3.6) subject to an absorbing boundary condition at $x = A$, i.e., subject to the condition $P(A, t) = 0$. The probability $G(t)$ that a diffusing particle has not been absorbed at time t is

$$G(t) = \int_{-\infty}^{A} P(x, t)\, dx = 2F\left(\frac{A}{(2\mathscr{D}t)^{1/2}}\right) - 1 \tag{3.7}$$

where without loss of generality we have taken $A > 0$ and where

$$F(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} \exp(-u^2/2)\, du \tag{3.8}$$

It is easy to see that asymptotically $G(t) = O(t^{-1/2})$, so that again one needs at least three independent diffusing particles to have a finite average time to absorption.

The continuum analog of (2.12) is

$$T_k = \int_0^\infty G^k(t)\, dt = C_k A^2/\mathscr{D} \tag{3.9}$$

where T_k is the mean time for the first of k particles to reach A and where C_k depends only on k. In particular, $C_3 = 0.7576$. One can make a connection between a continuous diffusion process and a discrete, nearest neighbor random walk by setting $a^2/2\mathscr{D} = \tau$, where τ is the time between steps and a is the lattice spacing. The number of steps to absorption implied by (3.9) is then $\langle n(\mathbf{l}|\mathbf{0})\rangle_k^{(1)} = 2C_k l^2$, where $l \equiv A/a$.

The above analysis has been restricted to random walks of finite variance. If the variance σ of the random walk is infinite, the results may be expected to be quite different. While we have not found it possible to develop a completely general theory in this case, one can investigate the behavior in one dimension for random walks in which the single-step transition probability $p(j) = |j|^{-\beta-1}$ for large j. In Appendix A we show that for $\beta = 2$, three random walkers are necessary to ensure finite first arrival time. If $1 < \beta < 2$, then the number of walkers needed is $k > \beta/(\beta - 1)$ and the number k is in fact infinite at $\beta = 1$.

3.2. Two Dimensions

For a symmetric random walk on a square lattice in two dimensions, it has been shown[4] that, to leading order in $(1 - z)$,

$$P(z; \mathbf{0}|\mathbf{0}) \sim -\frac{1}{2\pi\sigma_1\sigma_2} \ln(1 - z) \tag{3.10}$$

In two dimensions, as in one dimension, every random walker reaches any given site $\mathbf{l}$ with certainty, i.e., $f_{\mathbf{l}} = 1$.[1,2] Substitution of (3.10) in (2.9) and use of the fact that $P(1; \mathbf{0}|\mathbf{0}) - P(1; \mathbf{l}|\mathbf{0})$ is finite leads to

$$G(z; \mathbf{l}|\mathbf{0}) \sim \frac{2\pi\sigma_1\sigma_2[P(1; \mathbf{0}|\mathbf{0}) - P(1; \mathbf{l}|\mathbf{0})]}{(1 - z)\ln[1/(1 - z)]} \tag{3.11}$$

as $z \to 1$. The asymptotic form of G_n can then be obtained in the form[6]

$$G_n \sim \frac{2\pi\sigma_1\sigma_2[P(1; \mathbf{0}|\mathbf{0}) - P(1; \mathbf{l}|\mathbf{0})]}{\ln n} \tag{3.12}$$

It therefore follows that in two dimensions the mean time to absorption is infinite regardless of the number of random walkers that start from the origin. This conclusion is a consequence of the fact that the sum $\sum_{n=0}^{\infty} (\ln n)^{-k}$ diverges for all k.

3.3. Three Dimensions

In three dimensions it has been established[7] that around $z = 1$ the following expansion is valid:

$$\frac{P(z; \mathbf{l}|\mathbf{0})}{P(z; \mathbf{0}|\mathbf{0})} = f_{\mathbf{l}} - a_{\mathbf{l}}(1 - z)^{1/2} + \cdots \tag{3.13}$$

Here the factors $f_{\mathbf{l}}$ and $a_{\mathbf{l}}$ depend on the lattice structure. The probability $f_{\mathbf{l}}$ that a walker ever reaches site $\mathbf{l}$ is less than unity in three dimensions.[4,5] Using (3.13) in (2.9), we find that in the neighborhood of $z = 1$,

$$G(z; \mathbf{l}|\mathbf{0}) \sim \frac{a_{\mathbf{l}}}{f_{\mathbf{l}}} \frac{1}{(1 - z)^{1/2}} \tag{3.14}$$

This functional form is precisely the one obtained in (3.3) for the one-dimensional case. We thus again infer that

$$G_n \sim A_{\mathbf{l}}/\sqrt{n} \tag{3.15}$$

where $A_{\mathbf{l}}$ is independent of n. The mean time for the first walker to reach $\mathbf{l}$ is therefore finite if at least *three* walkers eventually reach $\mathbf{l}$.

3.4. Four Dimensions

In Appendix B we show that the following expansion is valid in four dimensions[3]:

$$\frac{P(z;\mathbf{l}|\mathbf{0})}{P(z;\mathbf{0}|\mathbf{0})} = f_1 + a_1(1-z)\ln(1-z) + \cdots \tag{3.16}$$

The functions f_1 and a_1 in (3.16) are independent of $(1-z)$, and f_1 is less than unity.[(5)] Substituting (3.16) in (2.9), we obtain

$$G(z;\mathbf{l}|\mathbf{0}) = (a_1/f_1)\ln(1-z) \tag{3.17}$$

Use of a Tauberian theorem[(6)] on Eq. (3.17) yields the asymptotic result

$$G_1 + G_2 + \cdots + G_n \sim \ln n \tag{3.18}$$

which implies that

$$G_n \sim 1/n \tag{3.19}$$

It thus follows via Eq. (2.12) that the first walker to reach $\mathbf{l}$ will do so in a finite time if at least *two* walkers eventually reach that site.

3.5. Dimensions Higher Than Four

In higher dimensions, we show in Appendix B that the ratio $P(z;\mathbf{l}|\mathbf{0})/P(z;\mathbf{0}|\mathbf{0})$ near $z=1$ has the expansion[3]

$$\frac{P(z;\mathbf{l}|\mathbf{0})}{P(z;\mathbf{0}|\mathbf{0})} = f_1 + a_1(1-z) + \cdots \tag{3.20}$$

where $f_1 < 1$,[(5)] and the specific forms of f_1 and a_1 depend on dimension and on lattice structure. It then follows from (2.9) that $G(1;\mathbf{l}|\mathbf{0})$ is finite and equal to a_1/f_1. Recalling that

$$G(1;\mathbf{l}|\mathbf{0}) = \sum_{n=0}^{\infty} G_n(\mathbf{l}|\mathbf{0}) = \langle n(\mathbf{l}|\mathbf{0})\rangle_1^{(D)}$$

is the mean time for a single random walker to reach $\mathbf{l}$ if it eventually does so, we thus conclude that in more than *four* dimensions every walker that reaches $\mathbf{l}$ does so in a finite time. The mean time for any of these walkers to arrive at $\mathbf{l}$ for large $|\mathbf{l}|$ is shown in Appendix B to be given by

$$\langle n(\mathbf{l}|\mathbf{0})\rangle_1^{(D)} = \frac{1}{D-4}\sum_{i=1}^{D}\frac{l_i^2}{\sigma_i^2}, \qquad D > 4 \tag{3.21}$$

[3] The analysis in this section is restricted to cases for which the only solution of the equation $\lambda(\boldsymbol{\theta}) = 1$ is $\boldsymbol{\theta} = \mathbf{0}$ with $\lambda(\boldsymbol{\theta})$ defined in (2.3) and $-\pi \leqslant \theta_i \leqslant \pi$.

where we have assumed the lattice to be simple cubic. If the variances $\sigma_i = \sigma$ are equal in all directions, then

$$\langle n(\mathbf{l}|\mathbf{0})\rangle_1^{(D)} = \frac{l^2}{(D-4)\sigma^2}, \qquad D > 4 \tag{3.22}$$

where $l^2 \equiv |\mathbf{l}^2|$. For nearest neighbor symmetric walks, (3.22) reduces to

$$\langle n(\mathbf{l}|\mathbf{0})\rangle_1^{(D)} = [D/(D-4)]l^2 \tag{3.23}$$

The formulas in Appendix B permit us to extend the above results to continuous dimensions. It is easily seen that the integral (B3) in fact converges for $D = 4 + \epsilon$ where ϵ is an arbitrarily small, positive number. Thus in all dimensions $D > 4$ only one random walker is required to obtain a finite mean time for arrival at $\mathbf{l}$, provided that the random walker reaches that point with unit probability. This behavior is reminiscent of the property of critical exponents, where for $D \leqslant 4$ the critical exponents depend on D, whereas for $D > 4$ the exponents take on their dimension-invariant classical values (see, e.g., Ref. 8).

4. DISCUSSION

The results obtained in Section 3 are summarized in Table I. The main features to note about the results are the following:

1. The number k of walkers that *must* reach site $\mathbf{l}$ to ensure that the first one to arrive there does so in a finite time is nonmonotonic with dimension D. The most remarkable aspect of this behavior is that whereas in one and in three or more dimensions the number of walkers k is finite, in two dimensions the mean time of first arrival at $\mathbf{l}$ is infinite, regardless of the number of walkers that arrive there. This is another example of the well-known fact that $D = 2$ is

Table I. First Passage Time $\langle n\rangle_k^{(D)}$ as a Function of k and D

D	f_l	k	$\langle n\rangle_k^{(D)}$
1	1	3	$c_1 l^2$
2	1	—	∞
3	<1	3	—
4	<1	2	—
>4	<1	1	$c_D l^2$

a "singular" dimension for a large number of random walk properties, due to the ubiquitous appearance of logarithmic factors. To understand this property, we note the following. In three or more dimensions, only a fraction $f_l < 1$ of walkers ever reach site $\mathbf{l}$, and only these are considered in the evaluation of the mean first passage time $\langle n \rangle_k^{(D)}$. We conjecture that the walkers that do reach site $\mathbf{l}$ in three and higher dimensions execute essentially a one-dimensional random walk, i.e., a walk confined to a narrow cylinder whose axis is defined by the origin and point $\mathbf{l}$. In two dimensions, however, *all* walkers reach the site $\mathbf{l}$ and those that sample an infinite two-dimensional region before reaching $\mathbf{l}$ take an infinitely long time to do so. Since these times are included in the first passage time for first arrival at $\mathbf{l}$, it is not possible to decrease this time of first arrival by increasing the number of walkers. According to our conjecture, those walkers whose contribution to $\langle n \rangle_k^{(2)}$ is infinite do not contribute to $\langle n \rangle_k^{(D \geqslant 3)}$, because they *never* arrive at $\mathbf{l}$.

2. The monotonic decrease from three to five dimensions in the number k of walkers necessary for a finite first arrival time can be explained as follows. In three and higher dimensions, we have conjectured that a walker that goes from the origin to site $\mathbf{l}$ does so along an essentially one-dimensional path with only very limited excursions into the other dimensions. These paths can be visualized as strictly one-dimensional walks with a finite probability per step of remaining at each site. As dimensionality increases, these pausing probabilities decrease since any excursion into other dimensions is more likely to cause the walker never to reach $\mathbf{l}$. Hence, as the dimensionality D increases, fewer walkers ever reach $\mathbf{l}$ (f_l decreases), but those that do arrive at $\mathbf{l}$ do so in a shorter time. This argument can only explain the trend but not the precise values of k.

3. The fact that in four and higher dimensions fewer walkers need to reach site $\mathbf{l}$ than in one dimension in order for the first to arrive at a finite expected time indicates a more "directed" walk in higher dimensions. One plausible conjecture to explain this behavior is that in higher dimensions walkers that stray too far in the direction opposite to $\mathbf{l}$, even in an essentially one-dimensional walk, never arrive at $\mathbf{l}$.

4. The arguments given above are reinforced by the fact that for $D > 4$, the mean first passage time $\langle n \rangle_1^{(D>4)}$ for a walker that does arrive at $\mathbf{l}$ is proportional to l^2, the square of the distance of site $\mathbf{l}$ from the origin. This behavior is consistent with that of one-dimensional walks. The coefficient of l^2 decreases with increasing dimensionality in accordance with the arguments in points 2 and 3 above. We have not calculated the mean times $\langle n \rangle_3^{(3)}$ and $\langle n \rangle_2^{(4)}$ in three and four dimensions, respectively, but conjecture that these are also proportional to l^2.

5. It is easy to show that *all* the moments $\langle n^s(\mathbf{l}|\mathbf{0}) \rangle_1^{(D>4)}$ are finite. It should be noted that all the moments $\langle n^s(\mathbf{l}|\mathbf{0}) \rangle_1^{(1)}$ are also finite for one random

walker in a one-dimensional random walk if there is a reflecting barrier at a site such that the origin of the walk is between the reflecting barrier and site **l**, whereas in the absence of such a barrier all these moments are infinite. This observation lends credence to our conjecture in point 3 of the directionality of the higher dimensional walks.

6. It is interesting to note that the smallest number of walkers with certain arrival at **l** required for finite mean first arrival time does not change for dimensions greater than four. This minimum number, $k = 1$, is thus a property that "sticks" at $D > 4$. Many of the previously studied lattice random walk properties pertaining to a single walker stick at $D > 2$. It is not clear whether the analogy of the dimensional dependence of this property with that found for critical exponents (see Section 3.5) has a deeper significance or is only coincidental. It might be of interest to pursue this point further.

7. In view of the fact that the likelihood of two or more walkers being at the same point at the same time rapidly decreases with increasing dimensionality, the assumption of *independent* walkers used in this paper is in fact no restriction for large D.

The analysis in this paper can rather easily be extended to continuous-time random walks,[4,9] provided the mean time between jumps as well as the variances of the single-step transition probabilities are finite.

We point out that our results are applicable to geometries not explicitly mentioned in the calculations. Thus, the mean time for the first of k walkers to reach a given point in one dimension is equal to the mean time for the first of k walkers to reach a given *line* (*plane*) in two (three) dimensions. This can be seen by projecting the higher dimensional walks onto the one-dimensional one. Similarly, a walk to a line in three dimensions has the same properties as that to a point in two dimensions.

One can also consider the first passage time for a set of k random walkers on a one-dimensional lattice to reach point **l** simultaneously. Foldes and Gabor[10] have shown that two random walkers eventually reach **l** simultaneously with certainty, while the probability of simultaneous arrival is < 1 for $k \geqslant 3$. These results are a consequence of the fact that the problem of the *simultaneous* arrival at **l** of k walkers in one dimension is equivalent to the problem of the return to the origin of a single walker in k dimensions with suitably defined transition probabilities. This correspondence allows us to conclude that the mean time for two walkers to first arrive at **l** simultaneously is infinite.

APPENDIX A. Asymptotic Behavior of $P(z; \mathbf{l}|0)$ for One-Dimensional Stable Laws

We will develop the asymptotic theory for a one-dimensional random walk of infinite variance in which the single-step transition probabilities are asymptotic to a stable law, $p(j) \sim |j|^{-\beta-1}$. The structure function $\lambda(\theta)$ behaves like

$$\lambda(\theta) = 1 - \alpha|\theta|^{\beta} + o(|\theta|^{\beta}) \tag{A.1}$$

in a neighborhood of $\theta = 0$. For z close to 1, $P(z; l|0)$ behaves like

$$\begin{aligned} P(z; l|0) &\sim \frac{1}{\pi}\int_0^{\infty} \frac{\cos l\theta}{1 - z + \alpha\theta^{\beta}}\, d\theta \\ &= \frac{1}{\pi\alpha^{1/\beta}}\frac{1}{(1-z)^{1-1/\beta}}\int_0^{\infty} \frac{\cos\{l[(1-z)/\alpha]^{1/\beta}v\}}{v^{\beta}+1}\, dv \end{aligned} \tag{A.2}$$

When this representation is substituted into Eq. (2.9), we are led to the problem of finding an expansion for the integral

$$I(z) = \int_0^{\infty} \frac{1 - \cos\{l[(1-z)/\alpha]^{1/\beta}v\}}{v^{\beta}+1}\, dv \tag{A.3}$$

valid for z close to 1. We will first assume that $1 < \beta \leqslant 2$. It is clear that $I(1) = 0$. The behavior of the integral near $z = 1$ is determined by the behavior of the integrand for large v. The form of the integrand given in Eq. (A.3) is somewhat inconvenient for the evaluation of $I(z)$ near $z = 1$. Hence we will write

$$\begin{aligned} I(z) &= \int_0^{\infty} [1 - \cos(\epsilon v)]\left\{\frac{1}{(v+1)^{\beta}}\right\} dv \\ &\quad + \int_0^{\infty} [1 - \cos(\epsilon v)]\left\{\frac{1}{v^{\beta}+1} - \frac{1}{(v+1)^{\beta}}\right\} dv \end{aligned} \tag{A.4}$$

where we have set

$$\epsilon = (l/\alpha^{1/\beta})(1-z)^{1/\beta} \tag{A.5}$$

The curly bracketed term in the second integrand in (A.4) is $O(v^{-\beta-1})$ for large v, while that in the first integrand is $O(v^{-\beta})$. This implies that the leading term in the expansion of $I(z)$ will come from the first integral, or

$$I(z) \sim \int_0^{\infty} \frac{1 - \cos(\epsilon v)}{(v+1)^{\beta}}\, dv \tag{A.6}$$

Using the representation

$$\frac{1}{s^{\beta}} = \frac{1}{\Gamma(\beta)}\int_0^{\infty} t^{\beta-1}e^{-st}\, dt \tag{A.7}$$

and interchanging orders of integration, we find that

$$
\begin{aligned}
I(z) &\sim \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-t}\, dt \int_0^\infty [1 - \cos(\epsilon v)] e^{-vt}\, dv \\
&= \frac{\epsilon^2}{\Gamma(\beta)} \int_0^\infty \frac{t^{\beta-2} e^{-t}}{t^2 + \epsilon^2}\, dt \\
&= \frac{\epsilon^{\beta-1}}{\Gamma(\beta)} \int_0^\infty \frac{u^{\beta-2} e^{-\epsilon t}}{u^2 + 1}\, du \\
&\sim \frac{\pi}{2\Gamma(\beta)} \frac{1}{|\cos(\pi\beta/2)|} \epsilon^{\beta-1} \\
&= \frac{\pi}{2\Gamma(\beta)|\cos(\pi\beta/2)|} \frac{l^{\beta-1}}{\alpha^{1-1/\beta}} (1 - z)^{1-1/\beta}
\end{aligned}
\tag{A.8}
$$

When we combine this with Eq. (2.9), we find that

$$
G(z; l|0) \sim \frac{\beta}{2\Gamma(\beta)} \frac{\sin(\pi/\beta)}{|\cos(\pi\beta/2)|} \frac{l^{\beta-1}}{\alpha^{1-1/\beta}} \frac{1}{(1 - z)^{1/\beta}}
\tag{A.9}
$$

But this implies that

$$
G_n \sim \frac{1}{2\Gamma(\beta)\Gamma(1 + 1/\beta)} \frac{\sin(\pi/\beta)}{|\cos(\pi\beta/2)|} \frac{l^{\beta-1}}{\alpha^{1-1/\beta}} \frac{1}{n^{1-1/\beta}}
\tag{A.10}
$$

for large n. For $\beta = 2$, this agrees with the result obtained in Eq. (3.5). When the parameter β is less than 2, we see that the mean time for the first of the walkers to reach a given point is finite provided that $k > \beta/(\beta - 1)$ independent random walkers are involved. As β approaches 1, the number of random walkers required to assure a finite mean tends to infinity, while for $\beta = 1$ the required number is infinite.

APPENDIX B. Details of the Expansion of $P(z; \mathbf{l}|\mathbf{0})$

To analyze the behavior of $P(z; \mathbf{l}|\mathbf{0})$ near $z = 1$, we can start from the formal decomposition

$$
P(z; \mathbf{l}|\mathbf{0}) = P(1; \mathbf{l}|\mathbf{0}) - \frac{1 - z}{(2\pi)^D} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\lambda(\boldsymbol{\theta}) \cos(\mathbf{l} \cdot \boldsymbol{\theta})\, d^D\boldsymbol{\theta}}{[1 - \lambda(\boldsymbol{\theta})][1 - z\lambda(\boldsymbol{\theta})]}
\tag{B.1}
$$

It is known that $P(1; \mathbf{l}|\mathbf{0})$ is finite in three or more dimensions[2-4] and that near $\boldsymbol{\theta} = \mathbf{0}$, $\lambda(\boldsymbol{\theta})$ can be expanded as

$$
\lambda(\boldsymbol{\theta}) \sim 1 - \frac{1}{2} \sum_{i=1}^{D} \sigma_i^2 \theta_i^2 + o(\theta^2)
\tag{B.2}
$$

By the assumptions made earlier, any singular behavior in the integral in Eq. (B.1) can be calculated from the behavior of the integrand near $\boldsymbol{\theta} = \mathbf{0}$. Since the Jacobian of the transformation to D-dimensional spherical coordinates contains a factor θ^{D-1}, we see that in a neighborhood of the origin the integrand in Eq. (B.1) goes as θ^{D-5}. Hence in five or more dimensions the integral converges when $z = 1$, allowing us to write $P(z; \mathbf{l}|\mathbf{0}) = P(1; \mathbf{l}|\mathbf{0}) - \beta_\mathbf{l}(1 - z)$, where

$$\beta_\mathbf{l} = \frac{1}{(2\pi)^D} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\lambda(\boldsymbol{\theta}) \cos(\mathbf{l}\cdot\boldsymbol{\theta})\, d^D\boldsymbol{\theta}}{[1 - \lambda(\boldsymbol{\theta})]^2} \tag{B.3}$$

It therefore follows that

$$\frac{P(z; \mathbf{l}|\mathbf{0})}{P(z; \mathbf{0}|\mathbf{0})} = f_\mathbf{l} + a_\mathbf{l}(1 - z) + \cdots \tag{B.4}$$

as $z \to 1$. Here $f_\mathbf{l}$ and $a_\mathbf{l}$ depend on dimension and on lattice structure.

In $D = 4$ dimensions the integrand in Eq. (B.1) is singular at $z = 1$ and we must analyze it more closely to determine the type of singularity. For simplicity we restrict ourselves to the case $\sigma_1{}^2 = \sigma_2{}^2 = \sigma_3{}^2 = \sigma_4{}^2 = \sigma^2$, although the more general case can be dealt with by essentially the same techniques. The singular behavior in z is determined by the analytic behavior of the integrand in a neighborhood of $\boldsymbol{\theta} = \mathbf{0}$. This implies that the singular behavior of the integral in Eq. (B.1) can be found from

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\lambda(\boldsymbol{\theta}) \cos(\mathbf{l}\cdot\boldsymbol{\theta})\, d^4\boldsymbol{\theta}}{[1 - \lambda(\boldsymbol{\theta})][1 - z\lambda(\boldsymbol{\theta})]} \sim \frac{2}{\sigma^2} \int \cdots \int \frac{d^4\boldsymbol{\theta}}{\theta^2(1 - z + \frac{1}{2}\sigma^2\theta^2)} \tag{B.5}$$

where $\theta^2 = \theta_1{}^2 + \theta_2{}^2 + \theta_3{}^2 + \theta_4{}^2$, and the limits of the integrals can be chosen arbitrarily, provided that they include the origin. For simplicity we choose a finite sphere of radius ρ and transform to four-dimensional spherical coordinates. Since the integrand is spherically symmetric, we have

$$\int \cdots \int \frac{\lambda(\boldsymbol{\theta}) \cos(\mathbf{l}\cdot\boldsymbol{\theta})\, d^4\boldsymbol{\theta}}{\theta^2(1 - z + \frac{1}{2}\sigma^2\theta^2)} = \frac{\pi^2}{2} \int_0^{\rho} \frac{\theta\, d\theta}{1 - z + \frac{1}{2}\sigma^2\theta^2} \tag{B.6}$$

The integral on the rhs is elementary, but all that we are really interested in is the behavior of the integral in the limit $z = 1$. By evaluating the integral we find that this behavior has the form $-(1/\sigma^2)\ln(1 - z)$, so that in four dimensions we find

$$P(z; \mathbf{l}|\mathbf{0})/P(z; \mathbf{0}|\mathbf{0}) = f_\mathbf{l} + a_\mathbf{l}(1 - z)\ln(1 - z) + \cdots \tag{B.7}$$

which is also the form when the $\sigma_i{}^2$ are not all equal.

We next consider the behavior of $G(1;\mathbf{l}|\mathbf{0})$ for $D > 4$ and for $\sum_i l_i^2/\sigma_i^2 \gg 1$. By substituting Eq. (B.4) into (2.9), we find that

$$G(1;\mathbf{l}|\mathbf{0}) = -\frac{a_1}{f_1} = \frac{d}{dz} \ln \frac{P(z;\mathbf{l}|\mathbf{0})}{P(z;\mathbf{0}|\mathbf{0})}\bigg|_{z=1} \tag{B.8}$$

The last expression can be evaluated in the large-$(\sum_i l_i^2/\sigma_i^2)$ limit by considering the behavior of the integrand in (B.3) near $\boldsymbol{\theta} = \mathbf{0}$. The simplest way to determine the asymptotics is to write

$$\begin{aligned} P(z;\mathbf{l}|\mathbf{0}) &\sim \frac{1}{(2\pi)^D} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\exp(-i\mathbf{l}\cdot\boldsymbol{\theta})\, d^D\boldsymbol{\theta}}{1 - z + \frac{1}{2}\sum \sigma_i^2 \theta_i^2} \\ &= \int_0^\infty e^{-t(1-z)} F_D(t)\, dt \end{aligned} \tag{B.9}$$

valid for z close to 1. In this representation, $F_D(t)$ is

$$F_D(t) = \frac{1}{(2\pi)^D} \prod_{j=1}^{D} \left\{ \exp\left[-il_j\theta - \frac{t\sigma_j^2}{2} \theta^2 \right] d\theta \right\} \tag{B.10}$$

The behavior of $P(z;\mathbf{l}|\mathbf{0})$ in the limit $z = 1$ can be determined from the large-t behavior of $F_D(t)$ by a well-known Abelian theorem for Laplace transforms.[11] For large t the limits of integration in Eq. (B.10) can be taken to be $\pm\infty$, so that

$$F_D(t) \sim \frac{1}{(2\pi)^{D/2}\sigma_1\sigma_2\cdots\sigma_D} \frac{1}{t^{D/2}} \exp\left(-\frac{1}{2t} \sum \frac{l_i^2}{\sigma_i^2} \right) \tag{B.11}$$

With this form for $F_D(t)$, we can evaluate the integral in Eq. (B.9) exactly, finding

$$P(z;\mathbf{l}|\mathbf{0}) \sim \left(\frac{1-z}{L} \right)^{(D/4)-(1/2)} K_{D/2-1}(2[L(1-z)]^{1/2}) \tag{B.12}$$

where $K_\upsilon(z)$ is a Bessel function and

$$L = \frac{1}{2} \sum_{i=1}^{D} \frac{l_i^2}{\sigma_i^2} \tag{B.13}$$

Since we are only interested in the limit $z = 1$, we must expand the Bessel function, using the expansion[12]

$$K_\upsilon(\epsilon) \sim \frac{\Gamma(\upsilon)}{(\epsilon/2)^\upsilon} \left[1 - \frac{1}{\upsilon - 1} \left(\frac{\epsilon}{2} \right)^2 + \cdots \right] \tag{B.14}$$

valid for small ϵ. When we substitute this result into Eq. (B.12), we find that

$$\ln P(z; \mathbf{l}|\mathbf{0}) \sim A - \frac{L}{\frac{1}{2}D - 2}(1 - z) + \cdots \tag{B.15}$$

where A is a constant. This result, together with Eq. (B.8), implies the validity of Eq. (3.21).

One can find the asymptotic behavior of $\langle n(\mathbf{l}|\mathbf{0})\rangle_k^{(D)}$ for $D = 3$ and $D = 4$, but the analysis is somewhat more involved and we have not done so as yet.

NOTE ADDED IN PROOF

P. Erdös and S. J. Taylor (*Acta Math. Acad. Sci. Hung.* **11**:231 (1960)) have discussed certain properties of intersections of random walk paths whose behavior also "sticks" at $D > 4$.

REFERENCES

1. G. Polya, *Math. Ann.* **84**:149 (1921).
2. E. W. Montroll, *SIAM J. Math.* **4**:241 (1956).
3. E. W. Montroll, *Proc. Symp. Appl. Math.* XVI:193 (1964).
4. E. W. Montroll and G. H. Weiss, *J. Math. Phys.* **6**:167 (1965).
5. E. W. Montroll, *J. Soc. Indust. Appl. Math.* **4**:241 (1956).
6. G. H. Hardy and J. E. Littlewood, *Proc. Lond. Math. Soc.* **30**:23 (1929); G. H. Hardy, *Divergent Series* (Oxford, 1949).
7. A. A. Maradudin, E. W. Montroll, G. H. Weiss, R. Hermann, and H. Milnes, *Acad. Roy. Belg. Cl. Sci. Mem.*, Vol. 14 (1960).
8. M. E. Fisher, *Rev. Mod. Phys.* **46**:597 (1974).
9. D. Bedeaux, K. Lakatos-Lindenberg, and K. E. Shuler, *J. Math. Phys.* **12**:2116 (1971).
10. S. Foldes and G. Gabor, *Discr. Math.* **24**:103 (1978).
11. G. Doetsch, *Theorie und Anwendungen der Laplace-Transformation* (Dover, 1945).
12. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (U.S. Government Printing Office, Washington, D.C., 1964).